

Inverse Problem in Quantitative Susceptibility Mapping

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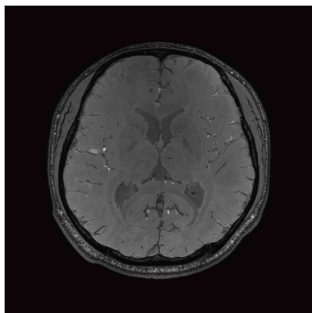
collaborated with Hyoung Suk Park, Shuai Wang, Yi Wang, and Jin Keun Seo

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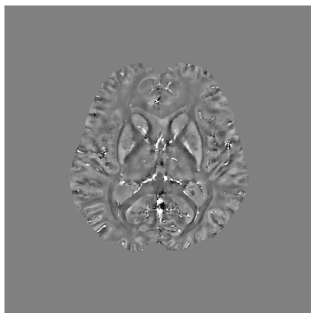
KSIAM 2014 Spring Conference

Quantitative Susceptibility Mapping (QSM)

- **Magnetic susceptibility** χ : an intrinsic property of the material relating the magnetization \mathbf{M} and the magnetic field \mathbf{H} via $\mathbf{M} = \chi\mathbf{H}$.
- **QSM**: aims to visualize **magnetic susceptibility** χ from MR data.



MR Magnitude Image



Susceptibility Image

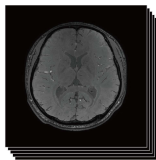
- **Applications**: Disease diagnosis (Parkinson's, Alzheimer, stroke, blood calcification, etc.)



MRI-Scanning
 k -space data

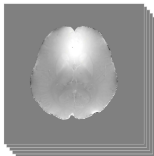


Inverse
Fourier Transform



Magnitude

Phase



Phase Unwrapping
Background Removal

RDF ψ



Dipole
Inversion

Reconstructed
Susceptibility χ



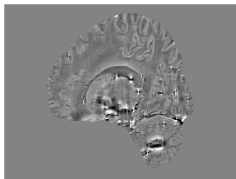
Inverse Problem of QSM

- Solve the deconvolution problem for χ :

$$\psi(\mathbf{x}) = \text{pv} \int_{\mathbb{R}^3} d(\mathbf{x} - \mathbf{y}) \chi(\mathbf{y}) d\mathbf{y} \quad d(\mathbf{x}) = \frac{2x_3^2 - x_1^2 - x_2^2}{4\pi|\mathbf{x}|^5}$$
$$\underbrace{\Psi(\boldsymbol{\xi})}_{\mathcal{F}(\psi)(\boldsymbol{\xi})} = \left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2} \right) \underbrace{\mathcal{X}(\boldsymbol{\xi})}_{\mathcal{F}(\chi)(\boldsymbol{\xi})} = \mathcal{D}(\boldsymbol{\xi}) \mathcal{X}(\boldsymbol{\xi}).$$

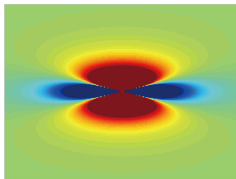
(pv: the principal value of the integral, \mathcal{F} : Fourier transform)

- Data: relative difference field (RDF) ψ (noisy)
- Integral kernel d : singular ($d(r\mathbf{x}) = r^{-3}d(\mathbf{x})$ for $r > 0$).



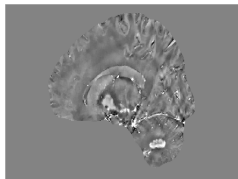
Data ψ (Noisy)

=



Unit dipole field d

*



Susceptibility χ

Challenging Issue

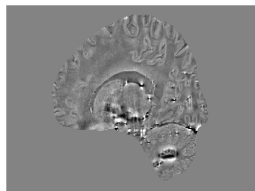
$$\psi(\mathbf{x}) = \text{pv} \int_{\mathbb{R}^3} d(\mathbf{x} - \mathbf{y}) \chi(\mathbf{y}) d\mathbf{y} \quad (\text{IP-I})$$

$$\underbrace{\Psi(\boldsymbol{\xi})}_{\mathcal{F}(\psi)(\boldsymbol{\xi})} = \underbrace{\left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2} \right)}_{\mathcal{D}(\boldsymbol{\xi})} \underbrace{\chi(\boldsymbol{\xi})}_{\mathcal{F}(\chi)(\boldsymbol{\xi})} \quad (\text{IP-F})$$

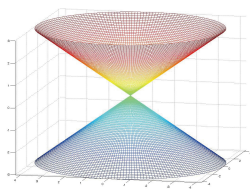
ill-posed since

$$\mathcal{D}(\boldsymbol{\xi}) = 0 \text{ in } \Gamma_0 = \{ \boldsymbol{\xi} \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 - 2\xi_3^2 = 0 \},$$

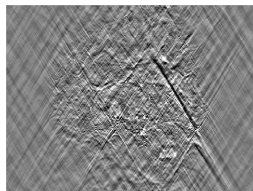
and this leads to the **streaking artifacts**.



Data ψ



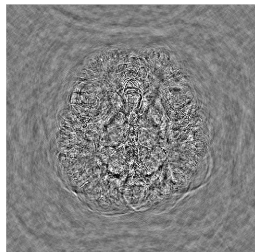
Γ_0 in Fourier domain



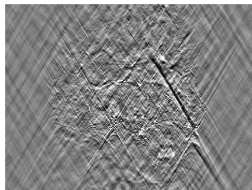
Reconstructed χ using (IP-F)

Observation & Goal

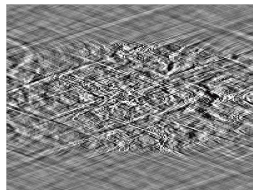
- **Streaking artifacts:** similar to the **wave propagation**.



Axial slice of χ



Sagittal slice of χ



Coronal slice of χ

- **Thorough understanding** of the structure of the equation and the solution is needed.
- **Goal:** provide **mathematical analysis on the inverse problem of QSM**.

Derivation of Forward Problem

Physical Background of QSM

We can obtain (IP-F)

$$\underbrace{\mathcal{F}\left(\frac{B_{\ell z} - B_0}{B_0}\right)}_{\mathcal{F}(\psi)(\boldsymbol{\xi}) = \Psi(\boldsymbol{\xi})}(\boldsymbol{\xi}) = \underbrace{\left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2}\right)}_{\mathcal{D}(\boldsymbol{\xi})} \underbrace{\frac{\mu_0}{B_0} \mathcal{F}(M_z)}_{\mathcal{F}(\chi)(\boldsymbol{\xi}) = \mathcal{X}(\boldsymbol{\xi})}(\boldsymbol{\xi}) \quad (\text{IP-F}).$$

- 1 by solving the magnetostatic Maxwell's equation for $\mathbf{B} = (B_x, B_y, B_z)$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{M}$$

with a given $\mathbf{M} = (0, 0, M_z)$ in the presence of $\mathbf{B}_0 = (0, 0, B_0)$ field,

- 2 applying the Lorentz Sphere correction to obtain $\mathbf{B}_\ell = (B_{\ell x}, B_{\ell y}, B_{\ell z})$:

$$\mathbf{B}_\ell(\mathbf{x}) = \mathbf{B}(\mathbf{x}) - \frac{2}{3} \mu_0 \mathbf{M}(\mathbf{x}) = \mathbf{B}_0(1 + \psi(\mathbf{x})),$$

- 3 and using the relation between χ and \mathbf{M} :

$$\chi(\mathbf{x}) = \frac{\mu_0}{B_0} M_z(\mathbf{x})$$

- Then the inverse Fourier transform on (IP-F) leads to the following PDE

$$\left(-\frac{1}{3}\Delta + \frac{\partial^2}{\partial x_3^2}\right)\chi = -\Delta\psi \quad (\text{IP-PDE}),$$

which means that if $\chi \in C_c^\infty$, then ψ can be written as

$$\psi(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \left(-\frac{1}{3}\Delta_{\mathbf{y}} + \frac{\partial^2}{\partial y_3^2}\right)\chi(\mathbf{y})d\mathbf{y}.$$

- Using the careful integration by parts, we obtain

$$\psi(\mathbf{x}) = \lim_{\varepsilon \searrow 0} \int_{|\mathbf{x} - \mathbf{y}| > \varepsilon} d(\mathbf{x} - \mathbf{y})\chi(\mathbf{y})d\mathbf{y} \quad d(\mathbf{x}) = \frac{2x_3^2 - x_1^2 - x_2^2}{4\pi|\mathbf{x}|^5}.$$

(The integral has to be understood as the principal value because $d(r\mathbf{x}) = r^{-3}d(\mathbf{x})$ for $r > 0$ with zero mean on S^2 .)

Our Main Results

- **Existence and Uniqueness Result**
- **Characterization of Streaking Artifacts**

Tool

- **Microlocal Analysis & Hörmander's Theory**

Both of the results are established for the first time.

Source of Error Propagation

- For a given measurement $\psi \in \mathcal{E}'$, we aim to obtain $\chi \in \mathcal{E}'$ using (IP-I) or (IP-F):

$$\psi(\mathbf{x}) = \lim_{\varepsilon \searrow 0} \int_{|\mathbf{x}-\mathbf{y}|>\varepsilon} d(\mathbf{x}-\mathbf{y})\chi(\mathbf{y})d\mathbf{y} \quad (\text{IP-I})$$

$$\Psi(\boldsymbol{\xi}) = \mathcal{D}(\boldsymbol{\xi})\mathcal{X}(\boldsymbol{\xi}) = \left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2} \right) \mathcal{X}(\boldsymbol{\xi}) \quad (\text{IP-F}).$$

(\mathcal{D}' : space of distributions, \mathcal{E}' : space of compactly supported distributions, \mathcal{S}' : space of tempered distributions)

- If $\psi \in \mathcal{E}'$ satisfies (IP-F) for some $\chi \in \mathcal{E}'$, then ψ must lie in

$$\mathcal{E}'_{\diamond} := \{u \in \mathcal{E}' : \widehat{u}(\boldsymbol{\xi})/P(\boldsymbol{\xi}) \text{ is bounded near } \Gamma_0\}.$$

Here, $P(\boldsymbol{\xi})$ is the polynomial defined as

$$P(\boldsymbol{\xi}) = \frac{4\pi^2}{3}(\xi_1^2 + \xi_2^2 - 2\xi_3^2).$$

Theorem (Existence and Uniqueness [J.K.Choi *et al.* 2014.])

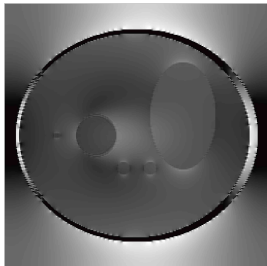
If $\psi \in \mathcal{E}'_{\diamond}$, we have the **unique** $\chi \in \mathcal{E}'$ satisfying (IP-F), and $\mathcal{X} = \mathcal{F}(\chi)$ can be represented as

$$\mathcal{X}(\xi) = \begin{cases} \frac{4\pi^2 |\xi|^2 \Psi(\xi)}{P(\xi)} & \text{if } \xi \notin \Gamma_0 \\ -\frac{9\xi_3}{4} \frac{\partial \Psi}{\partial \xi_3}(\xi) & \text{if } \xi \in \Gamma_0 \setminus \{0\}. \end{cases} \quad (\clubsuit)$$

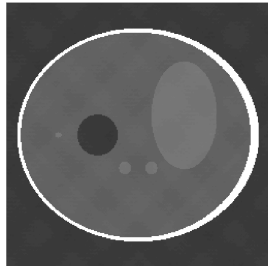
(Proof follows from Paley-Wiener-Schwartz theorem.)



Reference χ



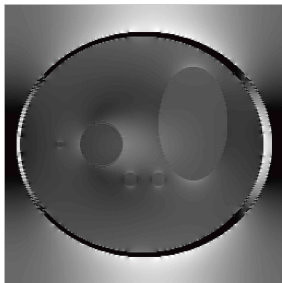
$\psi \in \mathcal{E}'_{\diamond}$



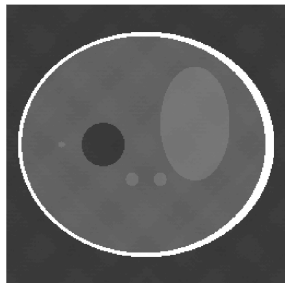
Reconstructed χ using (\clubsuit)



Reference x



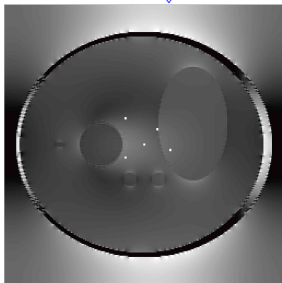
$\psi \in \mathcal{E}'_{\diamond}$



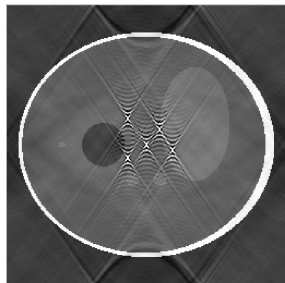
x w/o streaking artifacts



Reference x



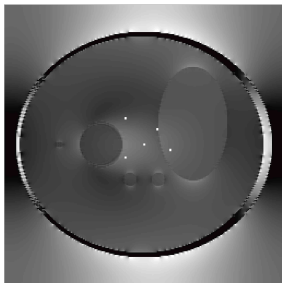
$\psi \in \mathcal{E}' \setminus \mathcal{E}'_{\diamond}$



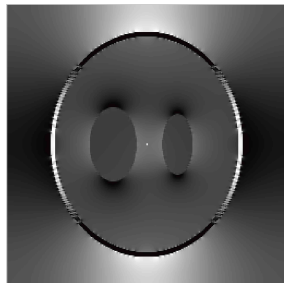
x with streaking artifacts



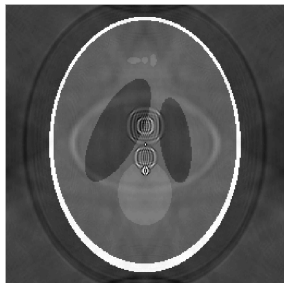
Axial slice of ψ



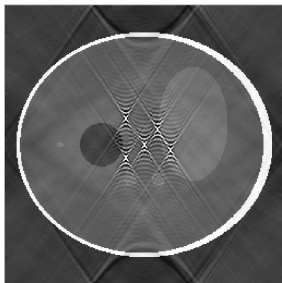
Sagittal slice of ψ



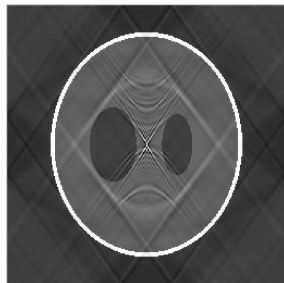
Coronal slice of ψ



Axial slice of χ



Sagittal slice of χ



Coronal slice of χ

Cause of Streaking Artifacts

- **Streaking artifacts: closely related with the PDE**

$$P(D)\chi = \left(-\frac{1}{3}\Delta + \frac{\partial^2}{\partial x_3^2} \right) \chi = -\Delta\psi \quad (\text{IP-PDE})$$

- The solution $\chi^\sharp \in \mathcal{D}'$ to (IP-PDE) is expressed as

$$\chi^\sharp(\mathbf{x}) = E * (-\Delta\psi)(\mathbf{x}) = - \int_{\mathbb{R}^3} E(\mathbf{x} - \mathbf{y}) \Delta_{\mathbf{y}} \psi(\mathbf{y}) d\mathbf{y} \quad \psi \in \mathcal{E}' \quad (\text{S-PDE})$$

where $E(\mathbf{x})$ is the fundamental solution of $P(D)$:

$$E(\mathbf{x}) = \begin{cases} \frac{3}{4\pi\sqrt{x_3^2 - 2(x_1^2 + x_2^2)}} & \text{if } 2(x_1^2 + x_2^2) < x_3^2 \\ 0 & \text{otherwise.} \end{cases}$$

$E(\mathbf{x})$ has the singular support along $\{\mathbf{x} \in \mathbb{R}^3 : 2(x_1^2 + x_2^2) = x_3^2\}$.

Theorem

For $\psi \in \mathcal{E}'$, let $\chi^\# \in \mathcal{D}'$ defined as (S-PDE). Then we have

$$\langle \chi^\#, \widehat{\varphi} \rangle = \lim_{\varepsilon \searrow 0} \frac{1}{2} \int_{\mathbb{R}^3} \left[\frac{\Psi(\xi - i\varepsilon e_3)}{\mathcal{D}(\xi - i\varepsilon e_3)} + \frac{\Psi(\xi + i\varepsilon e_3)}{\mathcal{D}(\xi + i\varepsilon e_3)} \right] \varphi(\xi) d\xi \quad (\spadesuit)$$

where $e_3 = (0, 0, 1)$ and $\varphi \in \mathcal{S}$.

- If $\psi \in \mathcal{E}'_\diamond$, then (\spadesuit) agrees with (\clubsuit) .
- If not, then

$$\mathcal{F}(\chi^\#)(\xi) = \frac{\Psi(\xi)}{\mathcal{D}(\xi)} \quad \text{for } \xi \in \mathbb{R}^3 \setminus \Gamma_0.$$

$$\underbrace{\Psi(\xi)}_{\mathcal{F}(\psi)(\xi)} = \underbrace{\left(\frac{1}{3} - \frac{\xi_3^2}{|\xi|^2} \right)}_{\mathcal{D}(\xi)} \underbrace{\chi(\xi)}_{\mathcal{F}(\chi)(\xi)} \quad (\text{IP-F}) \implies \chi(\xi) = \begin{cases} \frac{4\pi^2 |\xi|^2 \Psi(\xi)}{P(\xi)} & \text{if } \xi \notin \Gamma_0 \\ -\frac{9\xi_3}{4} \frac{\partial \Psi}{\partial \xi_3}(\xi) & \text{if } \xi \in \Gamma_0 \setminus \{0\}. \end{cases} \quad (\clubsuit)$$

$$P(D)\chi = \left(-\frac{1}{3}\Delta + \frac{\partial^2}{\partial x_3^2} \right) \chi = -\Delta \psi \quad (\text{IP-PDE}) \implies \chi^\# = E * (-\Delta \psi) \quad (\text{S-PDE})$$

Microlocal Analysis of Inverse Problem

Key Observation

To analyze the streaking artifacts in an image, **simultaneous concentration on both image and its Fourier transform** is crucial.

Definition

Wave front set of $u \in \mathcal{D}'$: a closed conic set in $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$

$$\text{WF}(u) = \{(x, \xi) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) : \xi \in \Sigma_x(u)\}.$$

$\xi \notin \Sigma_x(u) \iff \exists \varphi \in C_c^\infty$ with $\varphi(x) \neq 0$ and a conic nbd V of ξ s.t.

$$\sup_{\eta \in V} (1 + |\eta|)^N |\widehat{\varphi u}(\eta)| < \infty \quad \forall N \in \mathbb{N}.$$

If $(x, \xi) \in \text{WF}(u)$, then

- 1 $x \in \text{sing-supp}(u) \implies$ **location of singularity**
- 2 $\xi \in \Sigma_x(u) \implies$ **cause of singularity**

Theorem (Characterization of Artifacts [J.K.Choi *et al.* 2014.]

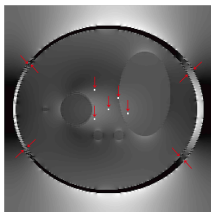
For $\psi \in \mathcal{E}'$, the wave front set of $\chi = \chi^\sharp$ satisfies

$$\text{WF}(\chi) \setminus \text{WF}(\psi) \subseteq \{(t\nabla P(\xi) + x, \xi) : \xi \in \Gamma_0 \setminus \{0\}, t \neq 0, (x, \xi) \in \text{WF}(\psi)\}$$

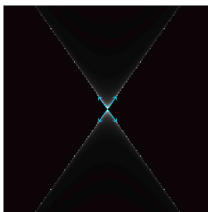
Moreover, if $(x, \xi) \in \text{WF}(\chi) \setminus \text{WF}(\psi)$, then

- 1 $\xi \in \Gamma_0$
- 2 for any open interval (a, b) containing 0 such that $\{(x + t\nabla P(\xi), \xi) : t \in (a, b)\} \cap \text{WF}(\psi) = \emptyset$, we have

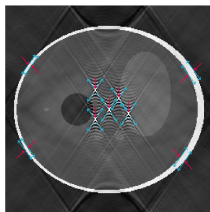
$$\{(x + t\nabla P(\xi), \xi) : t \in (a, b)\} \subseteq \text{WF}(\chi).$$



Simulated $\psi \in \mathcal{E}' \setminus \mathcal{E}'_0$



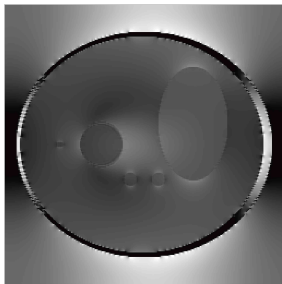
Fundamental solution E



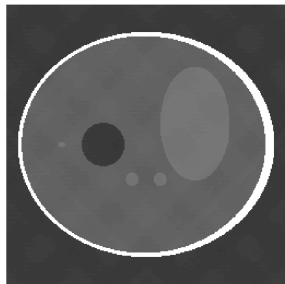
$\chi^\sharp = E * (-\Delta\psi)$



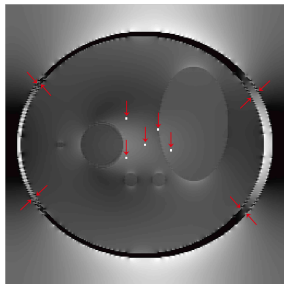
Reference x



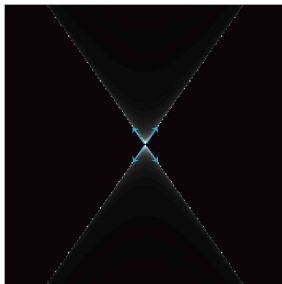
Simulated $\psi \in \mathcal{E}'_{\diamond}$



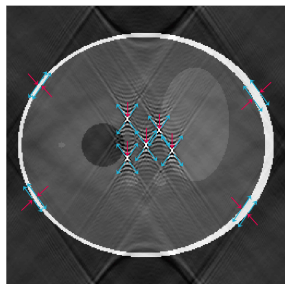
Reconstructed x w/o streaking artifacts



Simulated $\psi \in \mathcal{E}' \setminus \mathcal{E}'_{\diamond}$



Fundamental solution E



x with streaking artifacts

Remarks on Reconstruction Methods

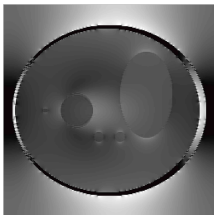
Direct Method

- Thresholded k -space division method (TKD):

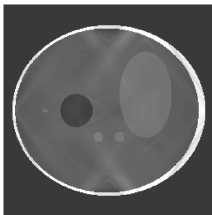
$$\mathcal{X}_{\hbar}(\xi) = \begin{cases} \frac{\Psi(\xi)}{\mathcal{D}(\xi)} & \text{if } |\mathcal{D}(\xi)| \geq \hbar \\ \text{sign}(\mathcal{D}(\xi)) \frac{\Psi(\xi)}{\hbar} & \text{if } |\mathcal{D}(\xi)| < \hbar, \end{cases} \quad \implies \chi_{\hbar} = \mathcal{F}^{-1}(\mathcal{X}_{\hbar}).$$

[K.Shmueli *et al.* 2009, S.Wharton *et al.* 2010]

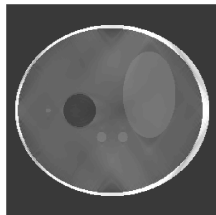
- **Straightforward** to compute
- **Additional streaking artifacts** (depending on $\hbar > 0$) even when $\psi \in \mathcal{E}'_{\diamond}$



Simulated $\psi \in \mathcal{E}'_{\diamond}$



χ_{\hbar} ($\hbar = 0.08$)



χ_{\hbar} ($\hbar = 0.16$)

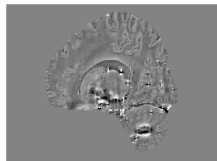
Bayesian Approach

- **Reduces streaking artifacts** using total variation and wavelet Φ :

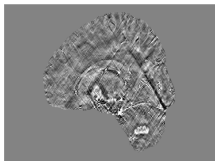
$$\chi = \arg \min \alpha \|\chi\|_{\text{TV}} + \beta \|\Phi\chi\|_1 + \frac{1}{2} \|\mathcal{D}\mathcal{F}(\chi) - \Psi\|_{L^2}^2 \quad \text{[TVL1L2]}$$

[B.Wu *et al.* 2012]

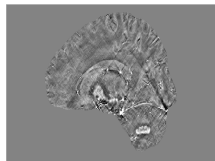
- May **lack realistic variations** (in the case of real measured data)



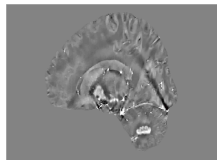
Measured $\psi \in \mathcal{E}'$



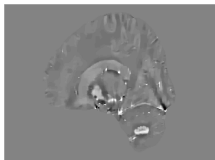
TKD ($h = 0.08$)



TKD ($h = 0.16$)



[TVL1L2] ($\alpha = \beta = 0.0005$)



[TVL1L2] ($\alpha = \beta = 0.002$)



[TVL1L2] ($\alpha = \beta = 0.008$)

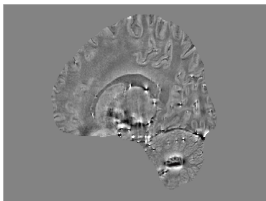
Morphology Enabled Bayesian Approach

- **Spatial priors** can be used to improve **[TVL1L2]**:

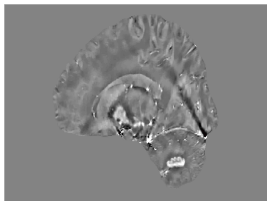
$$\chi = \arg \min \alpha \|\mathfrak{M} \nabla \chi\|_1 + \frac{1}{2} \|\mathfrak{W}(d * \chi - \psi)\|_{L^2}^2 \quad \text{[MEDI]}$$

[T.Liu *et al.* 2009] \mathfrak{M} : structural weighting matrix, \mathfrak{W} : noise weighting matrix

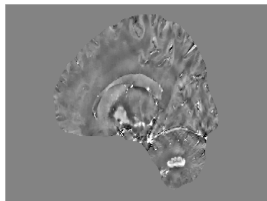
- Improve **morphological information**
- Obtain weighting matrices **empirically**



Measured $\psi \in \mathcal{E}'$



[TVL1L2] ($\alpha = \beta = 0.0005$)



[MEDI] ($\alpha = 0.0005$)

Conclusions

Conclusions

- We established the **theoretical ground** for the inverse problem of QSM; the **compatibility condition** of the data $\psi \in \mathcal{E}'$

$$\psi \in \mathcal{E}'_{\diamond} = \{u \in \mathcal{E}' : \widehat{u}(\xi)/P(\xi) \text{ is bounded near } \Gamma_0\}$$

is founded so that the inverse problem can be **solvable uniquely**.

- Any data that **violates the condition** will cause streaking artifacts **due to Γ_0** , which can be analyzed from the solution to the PDE:

$$\left(-\frac{1}{3}\Delta + \frac{\partial^2}{\partial x_3^2}\right) \chi = -\Delta \psi$$

using the **wave front set**.

- These theoretical studies will be useful to improve and develop QSM techniques so as to **reduce the streaking artifacts effectively**.

Special Thanks to

