Inverse Problem in Quantitative Susceptibility Mapping

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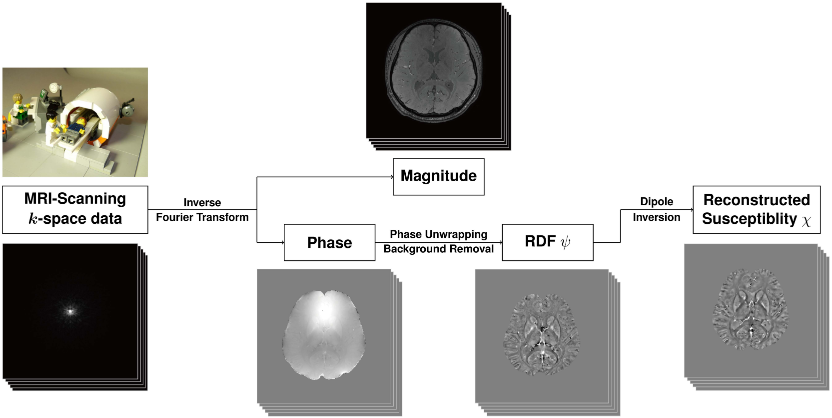
Quantitative Susceptibility Mapping (QSM)

QSM: aims to visualize **magnetic susceptibility** χ from MR data.



Inverse Problem of QSM

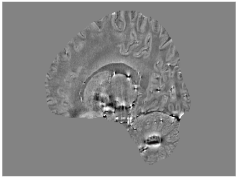
- Solve the deconvolution problem for χ:

$$\psi(\boldsymbol{x}) = \mathrm{pv}\int_{\mathbb{R}^3} d(\boldsymbol{x}-\boldsymbol{y})\chi(\boldsymbol{y})d\boldsymbol{y} \qquad d(\boldsymbol{x}) = \frac{2x_3^2 - x_1^2 - x_2^2}{4\pi|\boldsymbol{x}|^5}$$

$$\underbrace{\Psi(\boldsymbol{\xi})}_{\mathcal{F}(\psi)(\boldsymbol{\xi})} = \left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2}\right)\underbrace{\mathcal{X}(\boldsymbol{\xi})}_{\mathcal{F}(\chi)(\boldsymbol{\xi})} = \mathcal{D}(\boldsymbol{\xi})\mathcal{X}(\boldsymbol{\xi}).$$

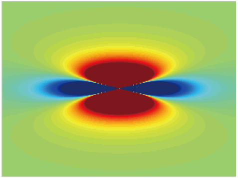
(pv: the principal value of the integral, $\mathcal{F}$: Fourier transform)

- Data: relative difference field (RDF) ψ (noisy)
- Integral kernel d: singular $(d(r\boldsymbol{x}) = r^{-3}d(\boldsymbol{x})$ for $r > 0)$.



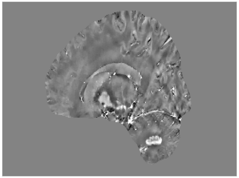
Data ψ (Noisy)

$=$



Unit dipole field d

$*$



Susceptibility χ

Challenging Issue

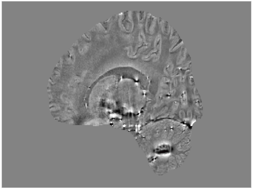
$$\psi(\boldsymbol{x}) = \mathrm{pv}\int_{\mathbb{R}^3} d(\boldsymbol{x}-\boldsymbol{y})\chi(\boldsymbol{y})d\boldsymbol{y} \qquad \text{(IP-I)}$$

$$\underbrace{\Psi(\boldsymbol{\xi})}_{\mathcal{F}(\psi)(\boldsymbol{\xi})} = \underbrace{\left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2}\right)}_{\mathcal{D}(\boldsymbol{\xi})} \underbrace{\mathcal{X}(\boldsymbol{\xi})}_{\mathcal{F}(\chi)(\boldsymbol{\xi})} \qquad \text{(IP-F)}$$

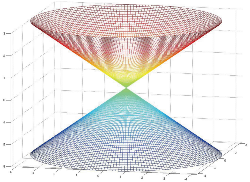
ill-posed since

$$\mathcal{D}(\boldsymbol{\xi}) = 0 \ \text{ in } \ \Gamma_0 = \left\{\boldsymbol{\xi} \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 - 2\xi_3^2 = 0\right\},$$

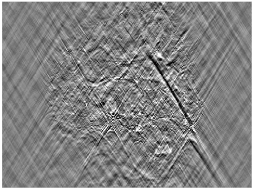
and this leads to the **streaking artifacts**.



Data ψ



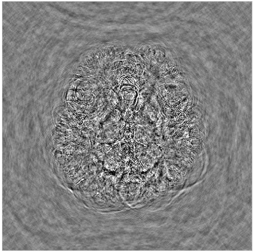
Γ_0 in Fourier domain



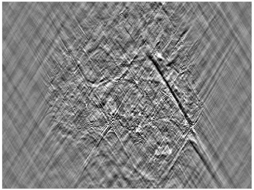
Reconstructed χ using (IP-F)

Observation & Goal

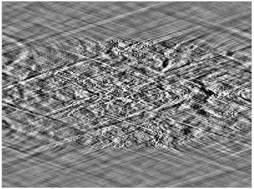
- **Streaking artifacts:** similar to the **wave propagation.**



Axial slice of χ



Sagittal slice of χ



Coronal slice of χ

- **Thorough understanding** of the structure of the equation and the solution is needed.
- **Goal:** provide **mathematical analysis on the inverse problem of QSM**.

Derivation of Forward Problem

Physical Background of QSM

We can obtain (IP-F)

$$\underbrace{\mathcal{F}\left(\frac{B_{\ell z}-B_0}{B_0}\right)(\boldsymbol{\xi})}_{\mathcal{F}(\psi)(\boldsymbol{\xi})=\Psi(\boldsymbol{\xi})} = \underbrace{\left(\frac{1}{3}-\frac{\xi_3^2}{|\boldsymbol{\xi}|^2}\right)}_{\mathcal{D}(\boldsymbol{\xi})} \underbrace{\frac{\mu_0}{B_0}\mathcal{F}(M_z)(\boldsymbol{\xi})}_{\mathcal{F}(\chi)(\boldsymbol{\xi})=\mathcal{X}(\boldsymbol{\xi})} \qquad \text{(IP-F)}.$$

1. by solving the magnetostatic Maxwell's equation for $\mathbf{B} = (B_x, B_y, B_z)$

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{M}$$

   with a given $\mathbf{M} = (0, 0, M_z)$ in the presence of $\mathbf{B}_0 = (0, 0, B_0)$ field,

2. applying the Lorentz Sphere correction to obtain $\mathbf{B}_\ell = (B_{\ell x}, B_{\ell y}, B_{\ell z})$:

$$\mathbf{B}_\ell(\boldsymbol{x}) = \mathbf{B}(\boldsymbol{x}) - \frac{2}{3}\mu_0 \mathbf{M}(\boldsymbol{x}) = \mathbf{B}_0(1+\psi(\boldsymbol{x})),$$

3. and using the relation between χ and $\mathbf{M}$:

$$\chi(\boldsymbol{x}) = \frac{\mu_0}{B_0} M_z(\boldsymbol{x})$$

- Then the inverse Fourier transform on (IP-F) leads to the following PDE

$$\left(-\frac{1}{3}\Delta + \frac{\partial^2}{\partial x_3^2}\right)\chi = -\Delta\psi \qquad \text{(IP-PDE)},$$

which means that if $\chi \in C_c^\infty$, then ψ can be written as

$$\psi(\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left(-\frac{1}{3}\Delta_{\boldsymbol{y}} + \frac{\partial^2}{\partial y_3^2}\right)\chi(\boldsymbol{y})d\boldsymbol{y}.$$

- Using the careful integration by parts, we obtain

$$\psi(\boldsymbol{x}) = \lim_{\varepsilon \searrow 0}\int_{|\boldsymbol{x}-\boldsymbol{y}|>\varepsilon} d(\boldsymbol{x}-\boldsymbol{y})\chi(\boldsymbol{y})d\boldsymbol{y} \qquad d(\boldsymbol{x}) = \frac{2x_3^2 - x_1^2 - x_2^2}{4\pi|\boldsymbol{x}|^5}.$$

(The integral has to be understood as the principal value because $d(r\boldsymbol{x}) = r^{-3}d(\boldsymbol{x})$ for $r > 0$ with zero mean on S^2.)

Inverse Problem-Existence and Uniqueness

Source of Error Propagation

- For a given measurement $\psi \in \mathcal{E}'$, we aim to obtain $\chi \in \mathcal{E}'$ using (IP-I) or (IP-F):

$$\psi(\boldsymbol{x}) = \lim_{\varepsilon \searrow 0} \int_{|\boldsymbol{x}-\boldsymbol{y}|>\varepsilon} d(\boldsymbol{x}-\boldsymbol{y})\chi(\boldsymbol{y})d\boldsymbol{y} \qquad \text{(IP-I)}$$

$$\Psi(\boldsymbol{\xi}) = \mathcal{D}(\boldsymbol{\xi})\mathcal{X}(\boldsymbol{\xi}) = \left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2}\right)\mathcal{X}(\boldsymbol{\xi}) \qquad \text{(IP-F).}$$

($\mathcal{D}'$: space of distributions, $\mathcal{E}'$: space of compactly supported distributions, $\mathcal{S}'$: space of tempered distributions)

- If $\psi \in \mathcal{E}'$ satisfies (IP-F) for some $\chi \in \mathcal{E}'$, then ψ must lie in

$$\mathcal{E}'_{\diamondsuit} := \left\{u \in \mathcal{E}' : \widehat{u}(\boldsymbol{\xi})/P(\boldsymbol{\xi}) \text{ is bounded near } \Gamma_0\right\}.$$

Here, $P(\boldsymbol{\xi})$ is the polynomial defined as

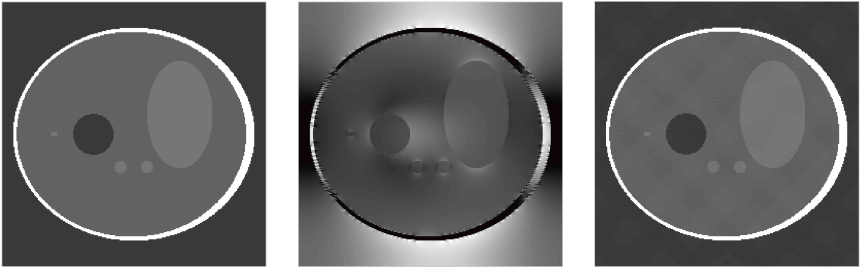
$$P(\boldsymbol{\xi}) = \frac{4\pi^2}{3}(\xi_1^2 + \xi_2^2 - 2\xi_3^2).$$

Theorem (Existence and Uniqueness [J.K.Choi *et al.* 2014.])

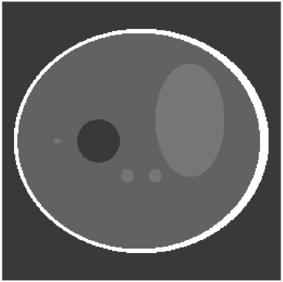
If $\psi \in \mathcal{E}'_{\diamondsuit}$, we have the **unique** $\chi \in \mathcal{E}'$ satisfying (IP-F), and $\mathcal{X} = \mathcal{F}(\chi)$ can be represented as

$$\mathcal{X}(\boldsymbol{\xi}) = \begin{cases} \dfrac{4\pi^2|\boldsymbol{\xi}|^2\Psi(\boldsymbol{\xi})}{P(\boldsymbol{\xi})} & \textit{if } \boldsymbol{\xi} \notin \Gamma_0 \\ -\dfrac{9\xi_3}{4}\dfrac{\partial\Psi}{\partial\xi_3}(\boldsymbol{\xi}) & \textit{if } \boldsymbol{\xi} \in \Gamma_0 \setminus \{\mathbf{0}\}. \end{cases} \qquad (\clubsuit)$$

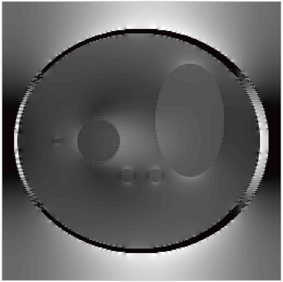
(Proof follows from Paley-Wiener-Schwartz theorem.)



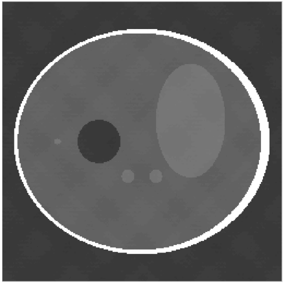
Reference χ | $\psi \in \mathcal{E}'_{\diamondsuit}$ | Reconstructed χ using $(\clubsuit)$



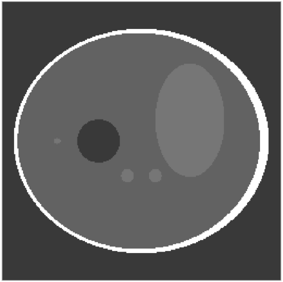
Reference χ



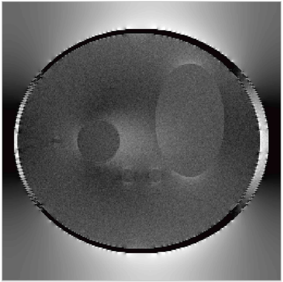
$\psi \in \mathcal{E}'_\diamond$



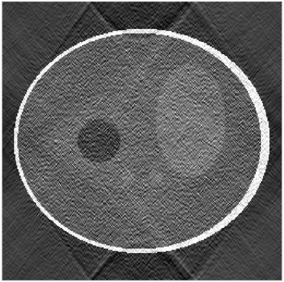
χ w/o streaking artifacts



Reference χ



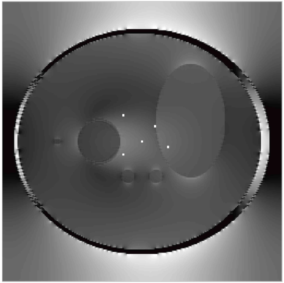
$\psi \in \mathcal{E}' \setminus \mathcal{E}'_\diamond$



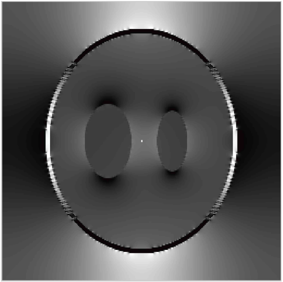
χ with streaking artifacts

Inverse Problem-Analysis on Streaking Artifacts

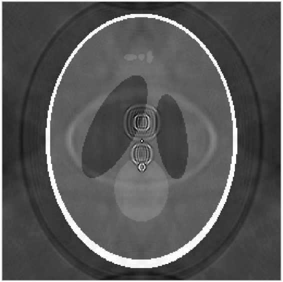
Axial slice of ψ



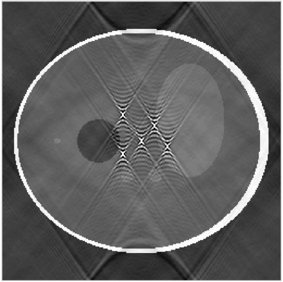
Sagittal slice of ψ



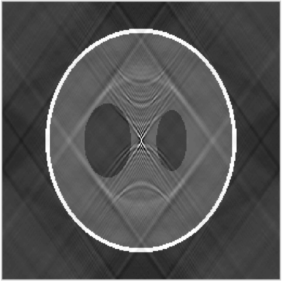
Coronal slice of ψ



Axial slice of χ



Sagittal slice of χ



Coronal slice of χ

Cause of Streaking Artifacts

- **Streaking artifacts: closely related with the PDE**

$$P(D)\chi = \left(-\frac{1}{3}\Delta + \frac{\partial^2}{\partial x_3^2}\right)\chi = -\Delta\psi \qquad \text{(IP-PDE)}$$

- The solution $\chi^\sharp \in \mathcal{D}'$ to (IP-PDE) is expressed as

$$\chi^\sharp(\boldsymbol{x}) = E * (-\Delta\psi)(\boldsymbol{x}) = -\int_{\mathbb{R}^3} E(\boldsymbol{x}-\boldsymbol{y})\Delta_{\boldsymbol{y}}\psi(\boldsymbol{y})d\boldsymbol{y} \qquad \psi \in \mathcal{E}' \qquad \text{(S-PDE)}$$

where $E(\boldsymbol{x})$ is the fundamental solution of $P(D)$:

$$E(\boldsymbol{x}) = \begin{cases} \dfrac{3}{4\pi\sqrt{x_3^2 - 2(x_1^2+x_2^2)}} & \text{if } 2(x_1^2+x_2^2) < x_3^2 \\ 0 & \text{otherwise.} \end{cases}$$

$E(\boldsymbol{x})$ has the singular support along $\{\boldsymbol{x} \in \mathbb{R}^3 : 2(x_1^2+x_2^2) = x_3^2\}$.

Theorem

For $\psi \in \mathcal{E}'$, let $\chi^\sharp \in \mathcal{D}'$ defined as (S-PDE). Then we have

$$\langle \chi^\sharp, \widehat{\varphi} \rangle = \lim_{\varepsilon \searrow 0} \frac{1}{2} \int_{\mathbb{R}^3} \left[\frac{\Psi(\boldsymbol{\xi} - i\varepsilon \boldsymbol{e}_3)}{\mathcal{D}(\boldsymbol{\xi} - i\varepsilon \boldsymbol{e}_3)} + \frac{\Psi(\boldsymbol{\xi} + i\varepsilon \boldsymbol{e}_3)}{\mathcal{D}(\boldsymbol{\xi} + i\varepsilon \boldsymbol{e}_3)} \right] \varphi(\boldsymbol{\xi}) d\boldsymbol{\xi} \quad (\spadesuit)$$

where $\boldsymbol{e}_3 = (0, 0, 1)$ and $\varphi \in \mathcal{S}$.

- If $\psi \in \mathcal{E}'_\diamondsuit$, then ($\spadesuit$) agrees with ($\clubsuit$).
- If not, then

$$\mathcal{F}(\chi^\sharp)(\boldsymbol{\xi}) = \frac{\Psi(\boldsymbol{\xi})}{\mathcal{D}(\boldsymbol{\xi})} \qquad \text{for} \quad \boldsymbol{\xi} \in \mathbb{R}^3 \setminus \Gamma_0.$$

$$\underbrace{\Psi(\boldsymbol{\xi})}_{\mathcal{F}(\psi)(\boldsymbol{\xi})} = \underbrace{\left(\frac{1}{3} - \frac{\xi_3^2}{|\boldsymbol{\xi}|^2} \right)}_{\mathcal{D}(\boldsymbol{\xi})} \underbrace{\mathcal{X}(\boldsymbol{\xi})}_{\mathcal{F}(\chi)(\boldsymbol{\xi})} \text{ (IP-F)} \implies \mathcal{X}(\boldsymbol{\xi}) = \begin{cases} \dfrac{4\pi^2 |\boldsymbol{\xi}|^2 \Psi(\boldsymbol{\xi})}{P(\boldsymbol{\xi})} & \text{if } \boldsymbol{\xi} \notin \Gamma_0 \\ -\dfrac{9\xi_3}{4} \dfrac{\partial \Psi}{\partial \xi_3}(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \in \Gamma_0 \setminus \{\mathbf{0}\}. \end{cases} \quad (\clubsuit)$$

$$P(D)\chi = \left(-\frac{1}{3}\Delta + \frac{\partial^2}{\partial x_3^2} \right) \chi = -\Delta \psi \text{ (IP-PDE)} \implies \chi^\sharp = E * (-\Delta \psi) \text{ (S-PDE)}$$

Microlocal Analysis of Inverse Problem

Key Observation

To analyze the streaking artifacts in an image, **simultaneous concentration on both image and its Fourier transform** is crucial.

Definition

Wave front set of $u \in \mathcal{D}'$: a closed conic set in $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\})$

$$\mathrm{WF}(u) = \left\{(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{\mathbf{0}\}) : \boldsymbol{\xi} \in \Sigma_{\boldsymbol{x}}(u)\right\}.$$

$\boldsymbol{\xi} \notin \Sigma_{\boldsymbol{x}}(u) \Longleftrightarrow \exists \varphi \in C_c^\infty$ with $\varphi(\boldsymbol{x}) \neq 0$ and a conic nbd V of $\boldsymbol{\xi}$ s.t.

$$\sup_{\boldsymbol{\eta} \in V} (1 + |\boldsymbol{\eta}|)^N |\widehat{\varphi u}(\boldsymbol{\eta})| < \infty \qquad \forall N \in \mathbb{N}.$$

If $(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathrm{WF}(u)$, then

1. $\boldsymbol{x} \in \text{sing-supp}(u) \Longrightarrow$ **location of singularity**
2. $\boldsymbol{\xi} \in \Sigma_{\boldsymbol{x}}(u) \Longrightarrow$ **cause of singularity**

Theorem (Characterization of Artifacts [J.K.Choi *et al.* 2014.])

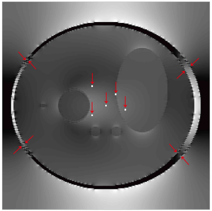
For $\psi \in \mathcal{E}'$, the wave front set of $\chi = \chi^\sharp$ satisfies

$$\mathrm{WF}(\chi) \setminus \mathrm{WF}(\psi) \subseteq \left\{ (t\nabla P(\boldsymbol{\xi}) + \boldsymbol{x}, \boldsymbol{\xi}) : \boldsymbol{\xi} \in \Gamma_0 \setminus \{\mathbf{0}\},\ t \neq 0,\ (\boldsymbol{x}, \boldsymbol{\xi}) \in \mathrm{WF}(\psi) \right\}$$

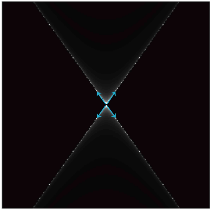
Moreover, if $(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathrm{WF}(\chi) \setminus \mathrm{WF}(\psi)$, then

1. $\boldsymbol{\xi} \in \Gamma_0$
2. for any open interval (a, b) containing 0 such that $\left\{ (\boldsymbol{x} + t\nabla P(\boldsymbol{\xi}), \boldsymbol{\xi}) : t \in (a, b) \right\} \cap \mathrm{WF}(\psi) = \emptyset$, we have

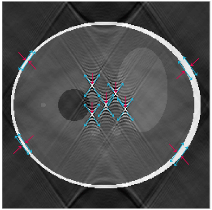
$$\left\{ (\boldsymbol{x} + t\nabla P(\boldsymbol{\xi}), \boldsymbol{\xi}) : t \in (a, b) \right\} \subseteq \mathrm{WF}(\chi).$$



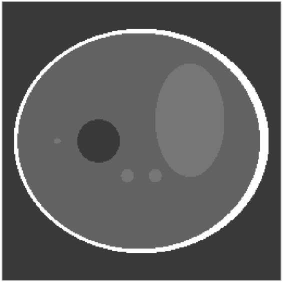
Simulated $\psi \in \mathcal{E}' \setminus \mathcal{E}'_\diamond$



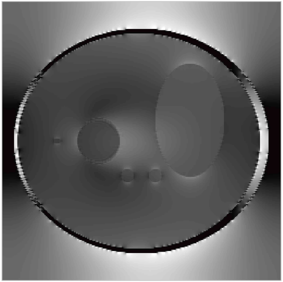
Fundamental solution E



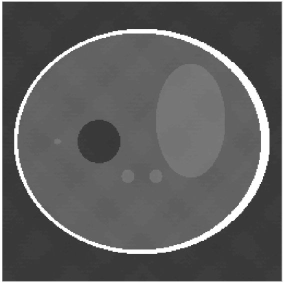
$\chi^\sharp = E * (-\Delta\psi)$



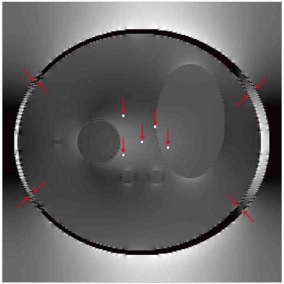
Reference χ



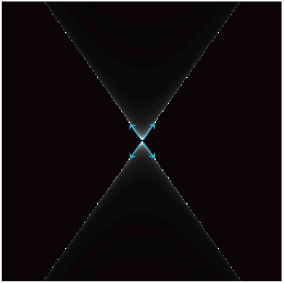
Simulated $\psi \in \mathcal{E}'_{\diamond}$



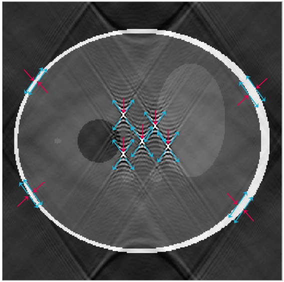
Reconstructed χ w/o streaking artifacts



Simulated $\psi \in \mathcal{E}' \setminus \mathcal{E}'_{\diamond}$



Fundamental solution E



χ with streaking artifacts

Remarks on Reconstruction Methods

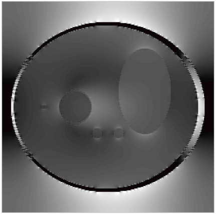
Direct Method

- Thresholded k-space division method (TKD):

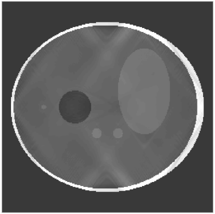
$$\mathcal{X}_\hbar(\boldsymbol{\xi}) = \begin{cases} \dfrac{\Psi(\boldsymbol{\xi})}{\mathcal{D}(\boldsymbol{\xi})} & \text{if } |\mathcal{D}(\boldsymbol{\xi})| \geq \hbar \\ \operatorname{sign}(\mathcal{D}(\boldsymbol{\xi}))\dfrac{\Psi(\boldsymbol{\xi})}{\hbar} & \text{if } |\mathcal{D}(\boldsymbol{\xi})| < \hbar, \end{cases} \implies \chi_\hbar = \mathcal{F}^{-1}(\mathcal{X}_\hbar).$$

[K.Shmueli *et al*. 2009, S.Wharton *et al*. 2010]

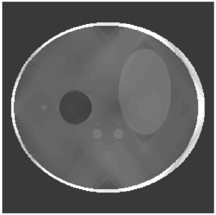
- **Straightforward** to compute
- **Additional streaking artifacts** (depending on $\hbar > 0$) even when $\psi \in \mathcal{E}'_\diamond$



Simulated $\psi \in \mathcal{E}'_\diamond$



$\chi_\hbar$ $(\hbar = 0.08)$



$\chi_\hbar$ $(\hbar = 0.16)$

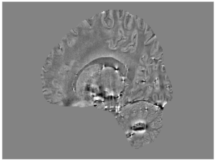
Bayesian Approach

- **Reduces streaking artifacts** using total variation and wavelet $\boldsymbol{\Phi}$:

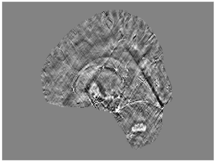
$$\chi = \arg\min \ \alpha\|\chi\|_{\mathrm{TV}} + \beta\|\boldsymbol{\Phi}\chi\|_1 + \frac{1}{2}\|\mathcal{D}\mathcal{F}(\chi) - \Psi\|_{L^2}^2 \qquad \textbf{[TVL1L2]}$$

[B.Wu *et al*. 2012]

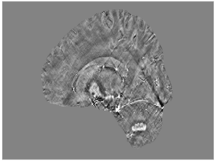
- May **lack realistic variations** (in the case of real measured data)



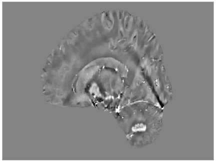
Measured $\psi \in \mathcal{E}'$



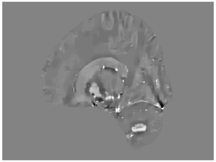
TKD $(\hbar = 0.08)$



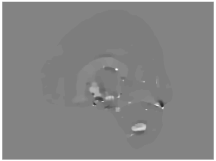
TKD $(\hbar = 0.16)$



[TVL1L2] $(\alpha = \beta = 0.0005)$



[TVL1L2] $(\alpha = \beta = 0.002)$



[TVL1L2] $(\alpha = \beta = 0.008)$

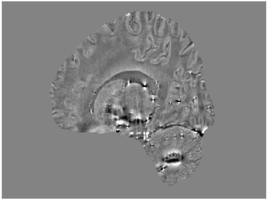
Morphology Enabled Bayesian Approach

- **Spatial priors** can be used to improve **[TVL1L2]**:

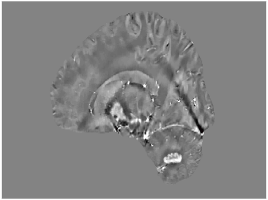
$$\min \ \alpha\|\mathfrak{M}\nabla\chi\|_1 + \frac{1}{2}\|\mathfrak{W}(d * \chi - \psi)\|_{L^2}^2 \qquad \textbf{[MEDI]}$$

[T.Liu *et al*. 2009] $\mathfrak{M}$: structural weighting matrix, $\mathfrak{W}$: noise weighting matrix

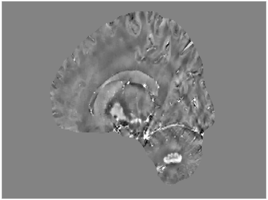
- Improve **morphological information**
- Obtain weighting matrices **empirically**



Measured $\psi \in \mathcal{E}'$



[TVL1L2] $(\alpha = \beta = 0.0005)$



[MEDI] $(\alpha = 0.0005)$

Conclusions

Conclusions

- We established the **theoretical ground** for the inverse problem of QSM; the **compatibility condition** of the data $\psi \in \mathcal{E}'$

$$\psi \in \mathcal{E}'_{\diamondsuit} = \left\{ u \in \mathcal{E}' : \widehat{u}(\boldsymbol{\xi})/P(\boldsymbol{\xi}) \text{ is bounded near } \Gamma_0 \right\}$$

  is founded so that the inverse problem can be **solvable uniquely.**

- Any data that **violates the condition** will cause streaking artifacts **due to** Γ_0, which can be analyzed from the solution to the PDE:

$$\left(-\frac{1}{3}\Delta + \frac{\partial^2}{\partial x_3^2} \right) \chi = -\Delta\psi$$

  using the **wave front set.**

- These theoretical studies will be useful to improve and develop QSM techniques so as to **reduce the streaking artifacts effectively.**

Thank you